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On the generalisation of the Kustaanheimo–Stiefel transformation

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Abstract. The Kustaanheimo-Stiefel and Levi-Cività transformations, used to regularise the three- and two-dimensional Kepler problem respectively, are generalised to the n-dimensional case. Explicit formulae are given for $n=2, 3, 5$, thus recovering in a more transparent way those given by Lambert and Kibler.

1. Introduction

The Hamiltonian vector field of the n-dimensional Kepler problem is not complete in the natural phase space, as collision orbits reach the singularity of the potential at finite times and with divergent velocity. The way of treating this pathology amounts to compactifying each cotangent space to the configuration manifold by adding the point at infinity. This is done by stereographic projection of the natural phase space $(\mathbf{R}^n - \{0\}) \times \mathbf{R}^n$ to $T^+S^n := T^*S^n - \{\text{null section}\}$ [1-3]. The Moser phase space T^+S^n obtained in this way turns out to be a co-adjoint orbit of the dynamical group $\mathrm{SO}(2, n+1)$ and the Hamiltonian flow is now regular and complete. A second way of regularising the problem entails a direct linearisation of the equations of motion. This dates back to Levi-Cività [4] for $n=2$ and to Kustaanheimo and Stiefel [5] for $n=3$.

In [6] a link between these two approaches is given via Clifford algebras and the Kustaanheimo-Stiefel transformation (KST) is generalised to n dimensions. In [7] the same generalisation is given but only for $n=5$; this is achieved via the (eight-dimensional) Cayley-Dickson algebra, so obtaining a quadratic non-bijective mapping $\mathbf{R}^8 \mapsto \mathbf{R}^5$ that generalises the analogous $\mathbf{R}^4 \mapsto \mathbf{R}^3$ of the KST. The link with the Kepler problem is recovered *a posteriori* in a pure computational way, showing that, as in the three-dimensional case, the KST transforms the motion equation of an eight-dimensional isotropic oscillator in that of the five-dimensional Kepler problem.

The aim of this paper is to show that the mapping found in [7] is a particular case of the general treatment given in [6], as is seen by making explicit the (somewhat abstract) statements of this last reference. Besides, in this way we reach a good understanding of the relation with the Kepler problem, which does not surprise us any longer.

In § 2 we review the contents of [6] and complete them with more explicit coordinate-dependent formulae. In § 3 we find explicitly the generalised KST for $n=5$ and show how the three cases considered in [6] (i.e. $n=2, 3, 5$) may be recovered from the general case: they differ only in that the real $\mathbf{R}$, the complex $\mathbf{C}$ and the quaternionic $\mathbf{H}$ numbers are to be employed. We stress finally that in the present approach one can naturally obtain the *complete* KST, i.e. the usual KST plus its cotangent lifting, thus generalising the formulae given by Kummer [2].

2. Regularisation of the Kepler problem

Let $i: T^+S^n \mapsto \mathrm{so}^*(2, n+1)$ be the immersion of the Moser phase space in the dual of the Lie algebra of $\mathrm{SO}(2, n+1)$, and let $\mathscr{O} = i(T^+S^n)$ be the corresponding co-adjoint orbit. Since $\mathrm{so}(2, n+1) = \mathrm{spin}(2, n+1)$, we can identify $\mathscr{O}$ with a co-adjoint orbit of the double covering $\mathrm{Spin}(2, n+1)$ of the dynamical group.

To get an explicit parametrisation of this orbit, we first consider the following.

Proposition 1. The Lie algebra $\mathrm{spin}(2, n+1)$ is a subalgebra of $\mathrm{su}(N, N)$, with $N = 2^{[n/2]}$.

For the proof see [6]. Here we give explicitly a basis of $\mathrm{spin}^*(2, n+1)$ through matrices that belong to $\mathrm{su}^*(N, N)$†. To this end, consider the n-dimensional Clifford algebra: its basis is generated by the $N \times N$ matrices Σ_h, $h = 1, \ldots, n$, that satisfy

$$\begin{aligned} &\Sigma_h \Sigma_k + \Sigma_k \Sigma_h = 2\delta_{hk} \mathbf{1} \\ &\Sigma_h^\dagger = \Sigma_h. \end{aligned} \tag{2.1}$$

These matrices always exist and can be easily constructed [8]. It is now a simple matter of calculation to verify that the following matrices:

$$\begin{aligned} &P_0 = \begin{pmatrix} 0 & \mathrm{i}\mathbf{1} \\ 0 & 0 \end{pmatrix} \qquad P_h = \begin{pmatrix} 0 & \mathrm{i}\Sigma_h \\ 0 & 0 \end{pmatrix} \qquad C_0 = \begin{pmatrix} 0 & 0 \\ -\mathrm{i}\mathbf{1} & 0 \end{pmatrix} \qquad C_k = \begin{pmatrix} 0 & 0 \\ \mathrm{i}\Sigma_k & 0 \end{pmatrix} \\ &D = \frac{1}{2}\begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \qquad J_{hk} = \frac{1}{2}\begin{pmatrix} -\Sigma_h \Sigma_k & 0 \\ 0 & -\Sigma_h \Sigma_k \end{pmatrix} \qquad J_{0h} = \frac{1}{2}\begin{pmatrix} -\Sigma_h & 0 \\ 0 & \Sigma_h \end{pmatrix} \end{aligned} \tag{2.2}$$

are a basis of $\mathrm{spin}^*(2, n+1)$. Here $J_{\mu\nu}$ are the generators of the Lorentz subgroup $\mathrm{SO}(1, n)$, P_μ and C_ν of the translations and conformal translations and D of the dilations. If $A \in \mathrm{spin}^*(2, n+1)$ and

$$\mathscr{E} = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \tag{2.3}$$

one verifies that $A^\dagger \mathscr{E} + \mathscr{E} A = 0$ and $\mathrm{Tr}\, A = 0$. Thus $A \in \mathrm{su}^*(N, N)$.

To parametrise the orbit $\mathscr{O} \subset \mathrm{spin}^*(2, n+1)$ we only need to fix one of its points Q_e. By identifying $\mathrm{spin}^*(2, n+1)$ with the Lie algebra of the conformal group of $\mathbf{R}^{1,n}$, one can choose Q_e to be a generator of a 'null translation' [3]; thus, for example,

$$Q_e = \begin{pmatrix} 0 & 0 & \mathrm{i}\mathbf{1} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.4}$$

† Being a simple group, we identify as usual the algebra with its dual by means of the trace form. They differ only by a sign in the compact part.

where each entry is a $(N/2)\times(N/2)$ matrix. The fundamental point is that we can take a 'square root' of the co-adjoint action of Spin$(2, n+1)$. The basic step is to notice that, if we define the $2N\times N/2$ matrix

$$\psi_e=\begin{pmatrix}\mathbf{1}\\0\\0\\0\end{pmatrix} \tag{2.5}$$

we can write

$$Q_e=i\psi_e\psi_e^{\dagger}\mathscr{E}. \tag{2.6}$$

Clearly any other $\psi_e'=\psi_e u$, for $u\in \mathrm{U}(N/2)$, will give the same Q_e. One takes care of such an ambiguity by taking the quotient under this right action of $\mathrm{U}(N/2)$. Since $\mathscr{E}g^{-1}=g^{\dagger}\mathscr{E}$, $\forall g\in \mathrm{Spin}(2, n+1)$, the co-adjoint action

$$Q_e\mapsto Q=gQ_eg^{-1} \tag{2.7}$$

is induced by the left action λ of Spin$(2, n+1)$ on ψ_e:

$$\psi_e\mapsto\psi=g\psi_e. \tag{2.8}$$

The orbit

$$\mathscr{T}=\{\psi|\psi=g\psi_e \bmod \mathrm{U}(N/2)\} \tag{2.9}$$

is contained in the symplectic manifold V of the complex $2N\times N/2$ matrices equipped with the natural symplectic form $\omega=\mathrm{d}\Theta$ where

$$\Theta=\tfrac{1}{2}\mathrm{i}\,\mathrm{Tr}(\psi^{\dagger}\mathscr{E}\,\mathrm{d}\psi-\mathrm{d}\psi^{\dagger}\mathscr{E}\psi). \tag{2.10}$$

If we give $\mathscr{T}$ the induced symplectic structure and $\mathcal{O}$ the Kirillov form, we have proposition 2.

Proposition 2. The map $J:\mathscr{T}\mapsto\mathcal{O}$ given by $\psi\mapsto J(\psi)=\mathrm{i}\psi\psi^{\dagger}\mathscr{E}$ is an equivariant moment map, i.e. a symplectomorphism that makes the diagram

$$\begin{array}{ccc}\mathscr{T} & \xrightarrow{\lambda} & \mathscr{T}\\ \downarrow{\scriptstyle J} & & \downarrow{\scriptstyle J}\\ \mathcal{O} & \xrightarrow{\mathrm{Ad}^*} & \mathcal{O}\end{array}$$

commutative, where Ad*is the co-adjoint action of Spin$(2, n+1)$.

For the proof see [6]. Let $\psi=\binom{z}{\mathrm{i}w}$ where z, w are $N\times N/2$ matrices. The moment map J becomes

$$J(\psi)=\begin{pmatrix}zw^{\dagger} & \mathrm{i}zz^{\dagger}\\ \mathrm{i}ww^{\dagger} & -wz^{\dagger}\end{pmatrix} \tag{2.11}$$

and we can decompose the subalgebra of translations as follows:

$$zz^{\dagger}=x_0\mathbf{1}+x_k\Sigma_k. \tag{2.12}$$

Remark 1. Consider the orbit $\mathscr{V} \subset V$ given still by (2.9) but with $g \in \mathrm{SU}(N, N)$. Obviously, owing to (2.5), we have that

$$\psi^{+}\mathscr{C}\psi = 0. \tag{2.13}$$

$\mathscr{V}$ can be obtained from V through symplectic reduction. Consider in fact the symplectic action of $\mathrm{U}(N/2)$ on V given by right multiplication and the corresponding moment map:

$$j = \mathrm{i}\psi^{+}\mathscr{C}\psi \qquad 0 \neq \psi \in V. \tag{2.14}$$

The procedure of the symplectic reduction gives the manifold $j^{-1}(0)/\mathrm{U}(N/2) = \mathscr{V}$. In general $\mathscr{T}$ is strictly contained in $\mathscr{V}$ and therefore this symplectic reduction does not give it. We shall see in § 3 that for the three cases considered in [7], i.e. for $n = 2, 3, 5$, the symplectic reduction is, on the contrary, sufficient.

Remark 2. Differentiating the left action λ we obtain

$$\mathrm{d}\psi/\mathrm{d}s = A\psi \qquad A \in \mathrm{spin}(2, n+1). \tag{2.15}$$

Choose A to be the generator of the SO(2) subgroup that appears as a factor in the maximal compact subgroup $\mathrm{SO}(2) \otimes \mathrm{SO}(n+1)$: it is the dual of $\frac{1}{2}(P_0 - C_0)$ and thus we obtain

$$\mathrm{d}z/\mathrm{d}s = \tfrac{1}{2}w \qquad \mathrm{d}w/\mathrm{d}s = -\tfrac{1}{2}z \tag{2.16}$$

i.e. the equations of motion of the isotropic oscillator.

We are now in the position of giving a general definition of the Kustaanheimo-Stiefel map $\mathscr{K}\mathscr{S}: \mathscr{T} \to T^*(\mathbf{R}^n - \{0\})$. Let $\pi: T^*(\mathbf{R}^n - \{0\}) \mapsto T^+S^n$ be the Moser regularising map: its inverse is given [1] by the extension of the stereographic projection followed by an interchanging between coordinates and momenta. Composing π with i we obtain the moment map: $T^*(\mathbf{R}^n - \{0\}) \mapsto \mathrm{so}^*(2, n+1)$ given explicitly [2, 3] by

$$\begin{aligned}
J_{hk} &= y_k x_h - y_h x_k \\
J_{0k} &= x y_k \\
D &= \langle x, y\rangle \\
P_0 &= x y^2 \\
P_k &= y^2 x_k - 2\langle x, y\rangle y_k \\
C_0 &= -x \\
C_k &= -x_k
\end{aligned} \tag{2.17}$$

where $x = (\Sigma_k x_k x_k)^{1/2}$. The Hamiltonian of the SO(2) subgroup of remark 2 is given by

$$K = \tfrac{1}{2}x(y^2 + 1) \tag{2.18}$$

i.e. by the Hamiltonian of the geodesic flow on S^n. Putting

$$x = q/K \qquad y = Kp \qquad E = -1/2K^2 \tag{2.19}$$

we obtain

$$\tfrac{1}{2}p^2 - 1/q = E \tag{2.20}$$

i.e. the Hamiltonian of the Kepler problem. (For a better understanding of this 'trick' see [3], where (2.19) is viewed as a canonical transformation in the enlarged phase space.) Let l be the symplectomorphism which makes the triangle in the diagram

$$\begin{tikzcd} T^*(\mathbf{R}^n-\{0\}) \arrow[r, "\pi"] & Y^-S^n \arrow[r, "i"] \arrow[d, "l"] & \mathcal{O} \\ & \mathcal{T} \arrow[ur, "J"'] & \end{tikzcd}$$

commutative. The map

$$\mathcal{KS}^{-1} := l \circ \pi$$

achieves in any dimension the target of regularising the Kepler problem by transforming the equivalent equations of motion of the geodesic flow on S^n into the form (2.16).

3. The case $n = 2, 3, 5$

Let us begin with $n = 3$, since in this case the above-defined $\mathcal{KS}$ is exactly the usual KST. We choose $\boldsymbol{\Sigma} = \boldsymbol{\sigma}$, where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.1}$$

are the well known Pauli matrices. Since $N = 2$, the relevant group is SU(2, 2), that being the twofold covering of SO(2, 4) is isomorphic to spin(2, 4). Thus V is the twistor space and $\mathcal{T} = \mathcal{V}$ is the orbit of the null twistors modulo phase transformations. Parametrise z in $\psi = \binom{z}{\mathrm{i}w}$ as

$$z = \sqrt{2}\begin{pmatrix} X_1 + \mathrm{i}X_2 \\ X_3 + \mathrm{i}X_4 \end{pmatrix} \qquad X_i \in \mathbf{R}. \tag{3.2}$$

From (2.12) the usual KST follows:

$$\begin{aligned} x_0 &= X_1^2 + X_2^2 + X_3^2 + X_4^2 \\ x_1 &= 2(X_1X_3 + X_2X_4) \\ x_2 &= 2(X_2X_3 - X_1X_4) \\ x_3 &= X_1^2 + X_2^2 - X_3^2 - X_4^2. \end{aligned} \tag{3.3}$$

Thus $x_0^2 = x_1^2 + x_2^2 + x_3^2$.

For $n = 2$ choose the basis: $\Sigma_1 = \sigma_1$, $\Sigma_2 = \sigma_3$, i.e. the one with real entries. The relevant group is still SU(2, 2) but now spin(2, 3) = Sp(4, $\mathbf{R}$) is a proper subgroup. The basis for spin*(2, 3) is given by (2.2) and one easily verifies that, restricting z and w to be real, the image of the moment map $J(\psi)$ given by (2.11) is just contained in spin*(2, 3). The orbit $\mathcal{V}$ is the same as in the preceding case, whereas the orbit $\mathcal{T} \subset \mathcal{V}$ is four dimensional. The symplectic reduction of remark 1 is empty, since the constraint (2.13) is identically satisfied, but we must divide out the $\mathbf{Z}_2$ subgroup of U(1). Parametrise z as

$$z = \sqrt{2}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \qquad X_i \in \mathbf{R}. \tag{3.4}$$

From (2.12) the usual Levi-Cività transformation follows:

$$\begin{aligned} x_0 &= X_1^2+X_2^2 \\ x_1 &= 2X_1X_2 \\ x_2 &= X_1^2-X_2^2. \end{aligned} \tag{3.5}$$

Thus $x_0^2=x_1^2+x_2^2$.

For $n=5$ choose the basis

$$\boldsymbol{\Sigma}=\begin{pmatrix} 0 & \mathrm{i}\boldsymbol{\sigma} \\ -\mathrm{i}\boldsymbol{\sigma} & 0 \end{pmatrix} \qquad \Sigma_4=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \Sigma_5=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.6}$$

The relevant group is now SU(4, 4) and Spin(2, 6) = SO*(8, $\mathbf{R}$) is a proper subgroup. Let z and w still be 2×1 matrices but with quaternionic entries, i.e. of the type

$$X = X_0\mathbf{1}+\mathrm{i}\boldsymbol{X}\cdot\boldsymbol{\sigma} \qquad X_0\in\mathbf{R} \qquad \boldsymbol{X}\in\mathbf{R}^3. \tag{3.7}$$

Thus one verifies that $J(\psi)$ (where $X^\dagger=X_0\mathbf{1}-\mathrm{i}\boldsymbol{X}\cdot\boldsymbol{\sigma}$) is contained in spin*(2, 6). Applying the symplectic reduction, ψ describes the orbit $\mathscr{V}$ under the action of SU(4, 4). Therefore $J(\psi)$, where ψ satisfies the constraint (2.13), describes the orbit $\mathscr{O}=T^+S^5$. Let us verify the dimensions: ψ, with z and w generic, describes a sixteen-dimensional manifold; being $\psi^\dagger\mathscr{E}\psi$ an imaginary quaternion, equation (2.13) gives a three-dimensional constraint; finally, we must divide out a unitary quaternion. Thus $\mathscr{O}$ is, as expected, ten dimensional. Parametrise z as

$$z=\sqrt{2}\begin{pmatrix} X_0+\mathrm{i}\boldsymbol{\sigma}\cdot\boldsymbol{X} \\ Y_0+\mathrm{i}\boldsymbol{\sigma}\cdot\boldsymbol{Y} \end{pmatrix}. \tag{3.8}$$

From (2.12) we obtain the generalised KST:

$$\begin{aligned} x_0 &= X_0^2+\boldsymbol{X}^2+Y_0^2+\boldsymbol{Y}^2 \\ \boldsymbol{x} &= 2(Y_0\boldsymbol{X}-X_0\boldsymbol{Y}+\boldsymbol{X}\times\boldsymbol{Y}) \\ x_4 &= X_0^2+\boldsymbol{X}^2-Y_0^2-\boldsymbol{Y}^2 \\ x_5 &= 2(X_0Y_0+\boldsymbol{X}\cdot\boldsymbol{Y}). \end{aligned} \tag{3.9}$$

Thus $x_0^2=\boldsymbol{x}^2+x_4^2+x_5^2$.

Summing up, in all three cases we pose $\psi=\binom{z}{\mathrm{i}w}$, where z and w are 2×1 matrices defined on $\mathbf{R}$, $\mathbf{C}$ and $\mathbf{H}$, respectively. Imposing the constraint $\psi^\dagger\mathscr{E}\psi=0$ and dividing out $\mathbf{Z}_2$, U(1) and SU(2) we obtain that $J(\psi)$ describes $\mathscr{O}=T^+S^n$ as a co-adjoint orbit of Spin(2, $n+1$): this is the link with the Kepler problem. Decomposing $zz^\dagger$ on the basis of the generators of the n-dimensional Clifford algebra we obtain the generalised KST. As a matter of principle, all of this can be extended to generic n, and in fact this has been done in § 2; but from the computational point of view we meet a serious obstruction since Spin(2, $n+1$) is not a classical group for $n\geqslant 6$.

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